

A NOTE ON THE DUALITY BETWEEN POISSON HOMOLOGY AND COHOMOLOGY

JIAFENG LÜ, XINGTING WANG AND GUANGBIN ZHUANG

ABSTRACT. For a Poisson algebra A , by studying its universal enveloping algebra A^{pe} , we prove a duality theorem between Poisson homology and cohomology of A .

1. INTRODUCTION

Poisson algebras, whose deformation are intimately related to some interesting non-commutative algebras (for example, Calabi-Yau algebras), have been intensively studied recently. It is known that for a lot of Poisson algebra A of dimension ℓ , the following duality holds

$$(1.1) \quad \mathrm{HP}_n(A) \cong \mathrm{HP}^{\ell-n}(A)$$

where $\mathrm{HP}_*(A)$ and $\mathrm{HP}^*(A)$ are Poisson homology and Poisson cohomology of A , respectively. In this paper, we prove a criterion for the duality (1.1) to hold. In fact, for a large class of Poisson algebras A of dimension ℓ , we are able to construct a right Poisson A -module ω_A such that

$$(1.2) \quad \mathrm{HP}_n(A, \omega_A) \cong \mathrm{HP}^{\ell-n}(A)$$

for any n . See Proposition 3.6 for the details. The duality (1.2) has been observed by Launois-Richard for a class of quadratic Poisson algebras [LR, Section 3.1]. The proof of our result will use the universal enveloping algebra A^{pe} of a Poisson algebra A .

Throughout this paper, let k denote a base field of characteristic 0. All algebras and tensor products are taken over k unless otherwise stated.

2. UNIVERSAL ENVELOPING ALGEBRAS OF POISSON ALGEBRAS

For a Poisson algebra A , one can define its universal enveloping algebra A^{pe} [Oh]. In [U], a constructive definition in terms of generators and relations are given. It is also observed, by several authors [LWZ, T], that A^{pe} is canonically isomorphic to $V(A, \Omega_A)$, where Ω_A is the Kähler differential of A and is equipped with the Lie-Rinehart algebra structure derived from the Poisson structure of A . For a detailed account of this isomorphism, one can refer to [LWZ, Proposition 5.7].

The construction of A^{pe} results in an algebra map $m : A \rightarrow A^{pe}$ and a Lie map $h : A \rightarrow A^{pe}$. The map m is injective and therefore we would simply consider A as a subalgebra of A^{pe} . Also, we will write h_a for $h(a)$.

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For a left Poisson module M with the bilinear map $\{, \} : A \otimes M \rightarrow M$, it naturally becomes left A^{pe} -module such that $\{a, m\}_M = h_a m$ for any $a \in A$ and $m \in M$. A similar statement holds for right Poisson modules as well. The following proposition is well known. See [U, Corollary 1].

Proposition 2.1. *The category of left (resp. right) Poisson modules over a Poisson algebra A is equivalent to the left (resp. right) module category over A^{pe} .*

Clearly, A can be viewed as a left A^{pe} -module as well as a right A^{pe} -module. In fact, as a left (resp. right) A^{pe} -module, A is isomorphic to the quotient of A^{pe} by the left (resp. right) ideal generated by elements h_a where $a \in A$.

The advantage of identifying A^{pe} with $V(A, \Omega_A)$ is that now we can use some standard results from the theory of Lie-Rinehart algebras. For example, the algebra $V(A, \Omega_A)$ carries a filtration which naturally passes to A^{pe} via the canonical isomorphism. The filtration $\{F_n\}_{n \geq 0}$ is such that $A \subset F_0 A^{pe}$ and $h_x \in F_1 A^{pe}$ for any $x \in A$. Now the theorem [Ri, Theorem 3.1] says the following.

Proposition 2.2. *Let A be a Poisson algebra. If the Kähler differential Ω_A is a projective A -module, then there is an A -algebra isomorphism*

$$(2.1) \quad S_A(\Omega_A) \cong \text{gr}_F A^{pe},$$

where $S_A(\Omega_A)$ is the symmetric A -algebra on Ω_A .

Here are some other consequences that will play essential roles in our paper.

Let A be a Poisson algebra. Consider the complex C_* where $C_n = 0$ for $n < 0$ and $C_n = A^{pe} \otimes_A \Omega_{A/k}^n$ for $n \geq 0$. Conventionally, we take $\Omega_{A/k}^0 = A$. The differential b is given by

$$\begin{aligned} b(a_0 \otimes da_1 da_2 \cdots da_n) &= \sum_{i=1}^n (-1)^{i+1} a_0 h_{a_i} \otimes da_1 da_2 \cdots \hat{da}_i \cdots da_n \\ &\quad + \sum_{1 \leq i < j \leq n} (-1)^{i+j} a_0 \otimes d\{a_i, a_j\} da_1 \cdots \hat{da}_i \cdots \hat{da}_j \cdots da_n. \end{aligned}$$

Proposition 2.3. *Let A be a Poisson algebra and suppose that Ω_A is projective over A . Then the complex C_* defined above is a projective resolution of A as a left A^{pe} -module.*

Proof. This is [Ri, Lemma 4.1]. □

Therefore, we get the following proposition, which to our knowledge is first explicitly spelled out in [H].

Proposition 2.4. *Let A be a Poisson algebra and suppose that Ω_A is projective over A . Let M be a left Poisson module and N a right Poisson module. Then*

$$(2.2) \quad \text{HP}^*(A, M) \cong \text{Ext}_{A^{pe}}^*(A, M),$$

and

$$(2.3) \quad \text{HP}_*(A, N) \cong \text{Tor}_*^{A^{pe}}(N, A),$$

where $\mathrm{HP}^*(A, M)$ is the Poisson cohomology with coefficients in M and $\mathrm{HP}_*(A, N)$ is the Poisson homology with coefficients in N .

In particular, if $M = A$ (resp. $N = A$), then $\mathrm{HP}^*(A, M)$ (resp. $\mathrm{HP}_*(A, N)$) is simply denoted by $\mathrm{HP}^*(A)$ (resp. $\mathrm{HP}_*(A)$) and called the Poisson cohomology (resp. Poisson homology) of A . However, in light of the previous proposition, one needs not to worry too much about those definitions, at least when Ω_A is projective over A , since they are just Ext and Tor which we assume most readers have a solid understanding on.

The rest of the section will be devoted to some details on the Lie-Rinehart algebra structure on Ω_A . First, let's recall the definition.

Definition 2.5. Let R be a commutative ring with identity, A a commutative R -algebra and L a Lie algebra over R . The pair (A, L) is called a *Lie-Rinehart algebra* over R if L is a left A -module and there is an *anchor map* $\rho : L \rightarrow \mathrm{Der}_R(A)$, which is an A -module and a Lie algebra morphism, such that the following relation is satisfied,

$$(2.4) \quad [\xi, a \cdot \zeta] = a \cdot [\xi, \zeta] + \rho(\xi)(a) \cdot \zeta,$$

for any $a \in A$ and $\xi, \zeta \in L$.

For simplicity, we will use $\xi(a)$ for $\rho(\xi)(a)$ where $\xi \in L$ and $a \in A$. In this paper, we would always assume that R is the ground field k . The following example is the one that we are mainly interested [H, Theorem 3.8].

Example 2.6. Let A be a Poisson algebra over k and Ω_A its Kähler differentials. Then the pair (A, Ω_A) becomes a Lie-Rinehart algebra over k where the Lie bracket on Ω_A is given by

$$(2.5) \quad [adf, bdg] = abd\{f, g\} + a\{f, b\}dg - b\{g, a\}df,$$

and the anchor map sends df to $\{f, \cdot\}$.

Now suppose that A is an affine Poisson algebra and that Ω_A is free over A of rank ℓ (which is equal to the Krull dimension of A) with a basis $\{dx_1, \dots, dx_\ell\}$. Then for any $y \in A$, there is a unique matrix $Y = (Y_{ij}) \in M_\ell(A)$ such that

$$[dy, dx_i] = \sum_j Y_{ij} dx_j$$

for any i . Define the trace of dy to be

$$(2.6) \quad \mathrm{tr}(dy) = \mathrm{tr} Y.$$

3. A DUALITY BETWEEN POISSON HOMOLOGY AND COHOMOLOGY

In this section, we are going to prove the main result of the paper.

Lemma 3.1. *Suppose that A is an affine Poisson algebra and that Ω_A is free over A of rank ℓ . Then $\mathrm{Ext}_{A^{pe}}^i(A, A^{pe}) = 0$ for $i \neq \ell$ and $\mathrm{Ext}_{A^{pe}}^\ell(A, A^{pe}) \neq 0$.*

Proof. By Proposition 2.2, $\text{gr } A^{pe} \cong A[y_1, \dots, y_\ell] := E$. Now we use the standard spectral sequence

$$\text{Ext}_E^*(A, E) \Rightarrow \text{Ext}_{A^{pe}}^*(A, A^{pe}).$$

Now the result follows from the fact that $\text{Ext}_E^i(A, E) = 0$ for $i \neq \ell$ and $\text{Ext}_E^\ell(A, E) \neq 0$. \square

Proposition 3.2. *Suppose that A is an affine Poisson algebra and that Ω_A is free over A of rank ℓ . Let ω_A be the right A^{pe} -module $\text{Ext}_{A^{pe}}^\ell(A, A^{pe})$. Then for any left A^{pe} -module (or equivalently, left Poisson module over A) N ,*

$$(3.1) \quad \text{Tor}_n^{A^{pe}}(\omega_A, N) \cong \text{Ext}_{A^{pe}}^{\ell-n}(A, N).$$

Proof. This is a consequence of Ischebeck's spectral sequence and Lemma 3.1. For the details, please see [K, Theorem 1.1, Corollary 1.4]. \square

Next we look closer at $\omega_A = \text{Ext}_{A^{pe}}^\ell(A, A^{pe})$.

Lemma 3.3. *Let A be as in Lemma 3.1 and let $\{dx_1, \dots, dx_\ell\}$ be a free A -basis of Ω_A . Then $\omega_A = \text{Ext}_{A^{pe}}^\ell(A, A^{pe})$ is isomorphic to A^{pe}/J as right A^{pe} -modules, where J is the right ideal generated by elements $h_{x_i} - \text{tr}(dx_i)$, $i = 1, 2, \dots, \ell$.*

Proof. The proof is just a direct calculation. Let C_* be the resolution as in Proposition 2.3. Then $\text{Ext}_{A^{pe}}^\ell(A, A^{pe})$ is the cokernel of the map

$$(3.2) \quad \text{Hom}_{A^{pe}}(C_{\ell-1}, A^{pe}) \xrightarrow{\partial} \text{Hom}_{A^{pe}}(C_\ell, A^{pe}).$$

Notice that C_ℓ is a free left A^{pe} -module of rank 1. In fact, given a free basis $\{dx_1, \dots, dx_\ell\}$ of Ω_A , C_ℓ is a free left A^{pe} -module with a basis $\{1 \otimes dx_1 dx_2 \dots dx_\ell\}$. Consequently, by sending $f \in \text{Hom}_{A^{pe}}(C_\ell, A^{pe})$ to $f(1 \otimes dx_1 dx_2 \dots dx_\ell) \in A^{pe}$, we can identify $\text{Hom}_{A^{pe}}(C_\ell, A^{pe})$ with A^{pe} as right A^{pe} -modules. Now a direct calculation shows that the image of ∂ in (3.2) is the right ideal of A^{pe} generated by elements of the form $h_{x_i} - \text{tr}(dx_i)$ where $i = 1, 2, \dots, \ell$. This completes the proof. \square

Question 3.4. Is there an automorphism ν on A^{pe} such that ν restricts to identity on A and $\omega_A \cong A^\nu$ as right A^{pe} -modules?

Remark 3.5. Let everything be as in Lemma 3.3, then the canonical map $A \rightarrow A^{pe}/J = \omega_A$ induced by the algebra map $m : A \rightarrow A^{pe}$ is an isomorphism of right A -modules. This is an easy consequence of Proposition 2.2. Under this identification, the right A^{pe} -module structure on $\omega_A = A$ is given by

$$(3.3) \quad mh_{x_i} = -\{x_i, m\} + m \text{tr}(dx_i),$$

where $m \in A$. Or, if we think ω_A as a right Poisson module over A with bilinear map $\{, \}_\omega : \omega_A \otimes A \rightarrow \omega_A$, the previous equation just translate into

$$(3.4) \quad \{m, x_i\}_\omega = -\{x_i, m\} + m \text{tr}(dx_i)$$

This equation has been observed by Launois-Richard for a class of quadratic Poisson algebras [LR].

Now we are ready to deliver the main result of this note.

Proposition 3.6. *Retain the notation in Lemma 3.1 and let $\{dx_1, \dots, dx_\ell\}$ be a free A -basis of Ω_A . Then*

$$(3.5) \quad \mathrm{HP}_n(A, \omega_A) \cong \mathrm{HP}^{\ell-n}(A)$$

for any n . Moreover, if $\mathrm{tr}(dx_i) = 0$ for any i , then

$$(3.6) \quad \mathrm{HP}_n(A) \cong \mathrm{HP}^{\ell-n}(A)$$

for any n .

Proof. The first statement is a consequence of Proposition 2.4 and Proposition 3.3. If $\mathrm{tr}(dx_i) = 0$ for any i , by Lemma 3.3, ω_A is isomorphic to A^{pe}/J where J is the right ideal generated by h_{x_i} . Hence ω_A is isomorphic to A as right A^{pe} -modules and therefore we have the second statement. \square

4. EXAMPLES

In this section, we look at some examples.

4.1. A Poisson algebra arising from a Calabi-Yau algebra. Let $A = k[x, y, z]$ with Poisson bracket given by

$$(4.1) \quad \{z, y\} = 2xz, \quad \{z, x\} = 0, \quad \{y, x\} = x^2.$$

This Poisson algebra is studied by Berger-Pichereau in [BP], whose deformation gives a type of Calabi-Yau algebra. In the same paper, the Poisson homology of A is also explicitly calculated [BP, Proposition 5.7]. Clearly, the Kähler differential Ω_A is free over A with a basis $\{dx, dy, dz\}$. Also,

$$\begin{aligned} [dz, dy] &= d\{z, y\} = 2x dz + 2z dx, \\ [dz, dx] &= d\{z, x\} = 0, \\ [dy, dx] &= d\{y, x\} = 2x dx. \end{aligned}$$

Consequently, $\mathrm{tr}(dx) = \mathrm{tr}(dy) = \mathrm{tr}(dz) = 0$ and therefore $\mathrm{HP}_n(A) \cong \mathrm{HP}^{3-n}(A)$. In fact, as pointed out in [BP], the Poisson algebra A is derived from the Poisson potential $\phi = -x^2z$ and thus the duality is automatic.

4.2. A class of quadratic Poisson algebras. In [LR], Launois-Richard studied the following Poisson algebra A . As an algebra, A is $k[X_1, X_2, \dots, X_\ell]$ and the Poisson bracket is given by

$$(4.2) \quad \{X_i, X_j\} = a_{ij}X_iX_j$$

where $(a_{ij}) \in M_n(k)$ is an antisymmetric matrix. A direct calculation shows that

$$(4.3) \quad \mathrm{tr}(dX_i) = \left(\sum_{j=1}^{\ell} a_{ij} \right) X_i$$

Therefore, as observed in Remark 3.5, ω_A is isomorphic to A as right A -modules and the right Poisson module structure on $\omega_A = A$ is given by

$$(4.4) \quad \{m, X_i\}_\omega = -\{X_i, m\} + \left(\sum_{j=1}^{\ell} a_{ij} \right) mX_i$$

for any $m \in \omega_A = A$. Notice that the equation (4.4) is exactly [LR, Section 3.1(6)]. Moreover, the twisted duality $\mathrm{HP}_n(A, \omega_A) \cong \mathrm{HP}^{\ell-n}(A)$ for this particular type of Poisson algebra is given in [LR, Theorem 3.4.2].

In fact, for this example, we can find an algebra automorphism ν on A^{pe} such that $\nu|_A = id$ and $\omega_A \cong A^\nu$. It is the unique map given by

$$\nu(X_i) = X_i, \quad \nu(h_{X_i}) = h_{X_i} + \left(\sum_{j=1}^{\ell} a_{ij} \right) X_i$$

for any i .

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LÜ: DEPARTMENT OF MATHEMATICS, ZHEJIANG NORMAL UNIVERSITY, JINHUA, CHINA

E-mail address: `jiafenglv@gmail.com`

WANG: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195, USA

E-mail address: `xingting@uw.edu`

ZHUANG: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES 90089-2532, USA

E-mail address: `gzhuang@usc.edu`